MATA22 Final Review

1. Vectors

Lecture Notes

- if $\vec{v}=r\vec{w}$ for some $r\in\mathbb{R}$, then $\vec{v}\parallel\vec{w}$
- $sp(\vec{v_1},\vec{v_2},\dots,\vec{v_k})=\{r_1\vec{v_1}+r_2\vec{v_2}+\dots+r_k\vec{v_k} \mid r_i\in\mathbb{R},\ 1\leq i\leq k\}$
- the span of two non-parallel vectors in $\mathbb{R}^2$ is $\mathbb{R}^2$
- <u>magnitude</u> or <u>norm</u> of $\vec{u}$ is $\|\vec{u}\|=\sqrt{u_1^2+u_2^2+\dots+u_n^2}$
- dot product: $\vec{v}\cdot\vec{w}=v_1w_1+v_2w_2+\dots+v_nw_n$
- angle between two nonzero vectors in $\mathbb{R}^2$ is $\arccos\left(\frac{v\cdot w}{\|v\|\|w\|}\right)$ where $\theta\in[0,\pi]$ $(0^\circ\sim 90^\circ)$
- <u>perpendicular</u> or <u>orthogonal</u>: $\vec{v}\cdot\vec{u}=0$
- distance: $\|\vec{v}-\vec{u}\|$
- C-S inequality: $|\vec{u}\cdot\vec{v}|\leq\|\vec{u}\|\|\vec{v}\|$
- Triangle Inequality: $\|\vec{u}+\vec{v}\|\leq\|\vec{u}\|+\|\vec{v}\|$
- orthogonal projection of $\vec{b}$ on $\vec{a}=\vec{p}=\mathrm{Proj}_a b=\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\cdot\vec{a}$
- vector component of $\vec{b}$ orthogonal to $\vec{a}=\vec{v}=\vec{b}-\vec{p}$

Assignments

① Prove that $\mathbf{u}\cdot\mathbf{v}=\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$ for all $\mathbf{u},\mathbf{v}\in\mathbf{R}^n$.

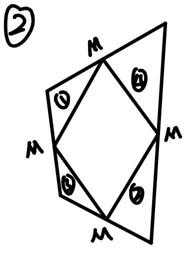
$\|\vec{u}+\vec{v}\|^2=\|\vec{u}\|^2+2\vec{u}\vec{v}+\|\vec{v}\|^2$ ①

$\|\vec{u}-\vec{v}\|^2=\|\vec{u}\|^2-2\vec{u}\vec{v}+\|\vec{v}\|^2$ ②

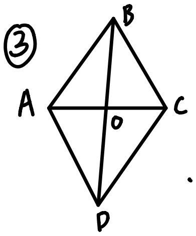
①-② $\|\vec{u}+\vec{v}\|^2-\|\vec{u}-\vec{v}\|^2=4\vec{u}\vec{v}$

$\therefore \frac{1}{4}\|\vec{u}+\vec{v}\|^2-\frac{1}{4}\|\vec{u}-\vec{v}\|^2=\vec{u}\vec{v}$

②



show ①=②, ③=④

③ 

$\vec{AC}, \vec{BD}$ bisect each other

$AB = \cdots AB \cdots BC$

$AB = AO + OB = rAC + kDB = r(AB+BC) + k(AB-BC) = rAB + rBC + kAB - kBC$

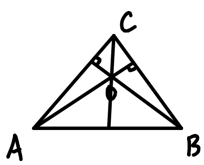
$\therefore AB = rAB + rBC + kAB - kBC$

$0 = rAB + rBC + kAB - kBC - AB$

$= AB(r+k-1) + BC(r-k)$

$\therefore \begin{cases} r+k-1=0 \\ r-k=0 \end{cases} \quad \begin{cases} r=\frac{1}{2} \\ k=\frac{1}{2} \end{cases}$

④ three altitudes of Δ meet.



Given: $AO \cdot BC = 0$

$BO \cdot AC = 0$

Show: $CO \cdot AB = 0$

$CO \cdot AB = CO \cdot (AC - BC)$

$= CO \cdot AC - CO \cdot BC$

$= (CB + BO) \cdot AC - (CA + AO) \cdot BC$ make $AO \cdot BC$, $BO \cdot AC$ appear

$= CB \cdot AC + 0 - CA \cdot BC - 0$

$= (-BC) \cdot (-CA) - CA \cdot BC$

$= CA \cdot BC - CA \cdot BC = 0$

⑤ $a_1, \ldots, a_n \in \mathbb{R}$

$$\frac{|a_1 + a_2 + \cdots + a_n|}{\sqrt{n}} \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

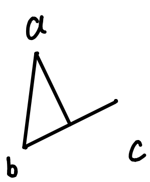
let $\vec{u} = [a_1, a_2, \ldots, a_n]$ and $\vec{v} = \underbrace{[1, 1, \ldots, 1]}_{n}$

by C-S $|u \cdot v| \leq \|u\| \|v\|$

$|a_1 + a_2 + \cdots + a_n| \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\,\sqrt{n}$

$$\therefore \frac{|a_1 + a_2 + \cdots + a_n|}{\sqrt{n}} \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

⑥ Shortest distant for a to the line $b \sim c$.



$\text{proj}_{bc} ba = \frac{ba \cdot bc}{\|bc\|^2} bc$

$v = ba - \text{proj}_{bc} ba$

$\therefore$ ans $= \|v\|$

⑦ through $(1,3)$ and $\parallel$ to $[2,-1]$

then $x = V + ta \quad t \in \mathbb{R}$, $V = [1,3] \quad a = [2,-1]$

$X = [X_1, X_2] = [1,3] + t[2,-1]$

⑧ Find equation of the plane with normal $[2,-1,5]$ and through $(7,3,-1)$

let $a=(a_1,a_2,a_3)$ on the plane then $[a_1-7, a_2-3, a_3+1]\cdot[2,-1,5]=0$

$\therefore 2a_1-14-a_2+3+5a_3+5=0$

$2a_1-a_2+5a_3=6$

then the equation of the plane are given by the point a is on the plane if $a_1=\frac{6+s-5t}{2}$, $a_2=s$, $a_3=t$.

⑨ find equation with three points $\underbrace{(1,3,2)}_{P}$ $\underbrace{(0,6,2)}_{Q}$ $\underbrace{(3,4,-3)}_{R}$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\vec{n}=\vec{PQ}=[-1,3,0]$

$\vec{v_2}=\vec{PR}=[2,1,-5]$

$$\begin{vmatrix} i & j & k \\ -1 & 3 & 0 \\ 2 & 1 & -5 \end{vmatrix} = i\begin{vmatrix} 3 & 0 \\ 1 & -5 \end{vmatrix} - j\begin{vmatrix} -1 & 0 \\ 2 & -5 \end{vmatrix} + k\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= \underbrace{-16}_{a}i \underbrace{-3}_{b}j \underbrace{-7}_{c}k$$

$\therefore -16(x-1)-3(y-3)-7(z-2)=0$

$-16x+16-3y+9-7z+14=0$

⑩ $P(6,-5,9)$ $Q(4,-1,5)$ $R(12,4,y)$

a) find all $y\in\mathbb{R}$ s.t PQR is a right angle triangle

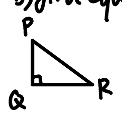
$\vec{PQ}=[-2,4,-4]$ $\vec{PR}=[6,9,y-9]$ $\vec{PQ}\cdot\vec{PR}=0 \Rightarrow -12+36+36-4y=0 \Rightarrow \underline{y=15}$

$\vec{QP}=[2,-4,4]$ $\vec{QR}=[8,5,y-5]$ $\vec{QP}\cdot\vec{QR}=6-20+4y-20=0 \Rightarrow \underline{y=6}$ $\quad 9-y\cdot 5-y$

$\vec{RP}=[-6,-9,9-y]$, $\vec{RQ}=[-8,-5,5-y]$ $\vec{RP}\cdot\vec{RQ}=48+45+45-9y-5y+y^2=y^2-14y+138$

$y^2-14y+138=0$ $\quad b^2-4ac=196-4\times1\times138<0$ $\therefore$ no root $\therefore y=15, y=6$.

b) find equation of the plane that contain PQR and Q is at the right angle



$\therefore \vec{QP}=[2,-4,4]$ $\vec{PR}=[6,9,-3]$, $\vec{QR}=[8,5,1]$. let $n=[a,b,c]$ s.t $\vec{QP}\cdot n=0$ $\vec{QR}\cdot n=0$ $\vec{PR}\cdot n=0$

$$\begin{cases} 2a-4b+4c=0 \\ 6a-9b-3c=0 \\ 8a+5b+c=0 \end{cases} \Rightarrow \begin{cases} a-2b+2c=0 \\ 21b-15c=0 \\ 21b-15c=0 \end{cases} = \begin{cases} a=-\frac{4}{7}c \\ b=\frac{5}{7}c \\ c=c \end{cases}$$ let $n=[-4,5,7]$

$a(x-x_0)+b(y-y_0)+c(z-z_0)$ using $n=[-4,5,7]$ and $Q=[4,-1,5)$

$-4(x-4)+5(y+1)+7(z-5)=0$

$\therefore -4x+5y+7z=14$.

2. Matrices & linear systems

Lecture Notes

- diagonal matrix: square $a_{ij}=0$ if $i\neq j$ $\begin{bmatrix} \ddots & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \ddots \end{bmatrix}$
- upper triangular matrix $\begin{bmatrix} \ddots & \# \\ 0 & \ddots \end{bmatrix}$
- identity matrix: square $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- lower triangular m
- $AB=\begin{bmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix}\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_k \\ | & | & & | \end{bmatrix}$
- symmetric matrix $A^T=A$
- skew symmetric matrix $A^T=-A$
- $Ax=b$ has a solution if b is in the span of col(A)
- Gauss' method with back substitution: get REF and back substitute
- Gauss-Jordan method: get RREF

$$\begin{array}{ccc} & I_m & \\ A_{m\times n} & \longrightarrow & B=EA \\ & \downarrow & \\ & E & \end{array}$$

- $A\sim I \Leftrightarrow Ax=b$ has a solution for $\forall b\in\mathbb{R}^n$

Assigments

① If $A_{n\times n}, B_{n\times n}$ prove $tr(AB)=tr(BA)$

$$tr(AB)=\sum_{i=1}^{n}(AB)_{ii}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}b_{ji}=\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ji}a_{ij}=\sum_{j=1}^{n}(BA)_{jj}=tr(BA)$$

② . Suppose that the matrices A and H below are row equivalent.

$$A=\begin{array}{c} \begin{bmatrix} 2 & 1 & x_3 & 1 & y_1 \\ 1 & -1 & 5 & 1 & y_2 \\ x_1 & 2 & 5 & -1 & y_3 \\ 0 & -1 & 2 & -1 & y_4 \\ 1 & x_2 & 3 & 1 & -2 \end{bmatrix} \\ \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \end{array} \sim \begin{array}{c} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} h_1 & h_2 & h_3 & h_4 & h_5 \end{array} \end{array} = H$$

Find the values of $x_1, x_2, x_3, y_1, y_2, y_3, y_4$,. Fully explain your reasoning.

$3h_1 - 2h_2 = h_3 \Leftrightarrow 3a_1 - 2a_2 = a_3 \Rightarrow \begin{bmatrix} x_3 \\ 5 \\ 5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3x_1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \\ 2x_2 \end{bmatrix}$

$\therefore \begin{cases} 3x_1 - 4 = 5 \\ 6-2 = x_3 \\ 3 - 2x_2 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 0 \\ x_3 = 4 \end{cases}$

$h_1 + 2h_2 - 3h_4 = h_5 \Rightarrow a_1 + 2a_2 - 3a_4 = a_5$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ -3 \\ -3 \\ 3 \end{bmatrix} \quad \therefore \begin{cases} y_1 = 1 \\ y_2 = -4 \\ y_3 = 10 \\ y_4 = 1 \end{cases}$$

③ $A_{n\times n}$ is skew symmetric matrix, $I_{n\times n}$ indentity matrix.

a) show that $A-I$ and $A+I$ are invertible

b) show that if $B = (I-A)(I+A)^{-1}$, then $B^TB = I$

a) since we know that A is invertible $\Leftrightarrow \vec{x} = \vec{0}$ is the only solution to $A\vec{x} = \vec{0}$

This is equivalenty proving $(A+I)\vec{x} = \vec{0} \Leftrightarrow \vec{x} = \vec{0}$

$\therefore$

$Ax = -x$

$(Ax)^T = -x^T$

$x^TA^T = -x^T$

$x^T - A = -x^T$

$x^TA = x^T$

$x^TAx = x^Tx$

$x^T(-x) = x^Tx$

$-\|x\|^2 = \|x\|^2$

$\therefore \|x\| = 0$

$x = 0$ ∎

b) Since $B = (I-A)(I+A)^{-1}$

$\therefore B^T = ((I-A)(I+A)^{-1})^T$

$= ((I+A)^{-1})^T(I-A)^T$

$= (I^T + A^T)^{-1}(I^T - A^T)$

$= (I-A)^{-1}(I+A)$

$B^TB = (I-A)^{-1}(I+A)(I-A)(I+A)^{-1}$

$= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1}$

$= I$ ∎

④ $A_{n\times n}$ $I_{n\times n}$ $O_{n\times n}$ zero matrix

$A^3-2A^2+3A-I=0$

show A is invertible and find A^{-1} in terms of A.

$A^3-2A^2+3A-I=0$ $\quad A^{-1}=B=A^2-2A+3I.$

$A^3-2A^2+3A=I$

$A\underbrace{(A^2-2A+3I)}_{B}=I$

⑤ Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5 \in \mathbf{R}^5$ satisfy the following two equations:

$$\begin{aligned} 2\mathbf{c}_1 + \mathbf{c}_2 - 3\mathbf{c}_4 - \mathbf{c}_5 &= \mathbf{0} \\ -\mathbf{c}_1 + 2\mathbf{c}_2 - \mathbf{c}_3 &= \mathbf{0} \end{aligned}$$

and let vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4$ be linearly independent.

If A is the matrix that has as its i^{th} column the vector $\mathbf{c}_i$ for $i = 1, 2, 3, 4, 5$, then find the row reduced echelon form of A. Justify your answer.

Let $A \sim H$ in RREF

$\therefore A\vec{x}=0 \Leftrightarrow H\vec{x}=0$

$\because 2c_1+c_2-3c_4-c_5=0$

$\therefore A\vec{v}=0$ where $\vec{v}=\begin{bmatrix}2\\1\\0\\-3\\-5\end{bmatrix}$

$\therefore H\vec{v}=0$ where $\vec{v}=\begin{bmatrix}2\\1\\0\\-3\\-5\end{bmatrix}$

and $h_5=2h_1+h_2-3h_4$

$h_3=2h_2-h_1$

$$\therefore \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \end{array}$$

linearly independ $\Rightarrow$ pivot in column.

3. Homogeneous System, Subspaces, Bases and Rank

Lecture Notes

- homogeneous system is consistent and is $A\vec{x}=\vec{0}$
- if the solution is $\vec{x}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \Rightarrow$ trivial solution
- if $A\sim H$ and H has pivots in every column $\Rightarrow Ax=0$ has the trivial solution $\Leftrightarrow$ invertible
- $null(A)=\{x\in\mathbb{R}^n \mid A\vec{x}=0\}$
- if $r_1\vec{v_1}+r_2\vec{v_2}+\dots+r_n\vec{v_n}=0$ has one solution $\Rightarrow v_1,v_2\dots,v_n$ are linearly independent
- if $r_1\vec{v_1}+r_2\vec{v_2}+\dots+r_n\vec{v_n}=0$ has more than one solution $\Rightarrow v_1,v_2\dots,v_n$ are linearly dependent
- Two vector in $\mathbb{R}^n$ are linearly indep. if they are nonzero and nonparallel.
- Let $W\subset\mathbb{R}^n$
  1. W is nonempty
  2. if $u,v\in W$, $u+w\in W$ (closure under vector addition)
  3. if $u\in W$, $r\in\mathbb{R}$, $ru\in W$ (closure under scalar mult.)

  then W is a subspace of $\mathbb{R}^n$

DEFINITION: Let W be a subspace of $\mathbb{R}^n$. If $\mathcal{B}=\{\mathbf{b}_1,\mathbf{b}_2,\cdots,\mathbf{b}_k\}$ is a subset of W, then we say that $\mathcal{B}$ is a basis for W if **every** vector in W can be written **uniquely** as a linear combination of the vectors in $\mathcal{B}$. The plural for the word basis is "bases".

THEOREM: Let A be an $n\times n$ matrix. The following are equivalent:

1. The linear system $A\mathbf{x}=\mathbf{b}$ has a unique solution for each $\mathbf{b}\in\mathbb{R}^n$.
2. The matrix A is row equivalent to the $n\times n$ identity, I.
3. The matrix A is invertible.
4. The column vectors of A form a basis for $\mathbb{R}^n$.
5. $rank(A)=dim(A)$
6. $det(A)\neq 0$

- $rank(A)+nullity(A)=n=dim(A)$

Assigments

① u is a nonzero column vector $\in \mathbb{R}^m$ and v is a colum vector $\mathbb{R}^n$

$A = uv^T$

a) show $col(A) = span(u)$ and $row(A) = span(v)$

b) nullity of A

c) show $null(A)$ contains all the vector orthogonal to v

a) $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $A = uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} [v_1 \ v_2 \ldots v_k] = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & & & \\ u_mv_1 & u_mv_2 & \cdots & u_mv_n \end{bmatrix}$

$= \begin{bmatrix} | & | & & | \\ uv_1 & uv_2 & \cdots & uv_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} u_1v \\ u_2v \\ \vdots \\ u_mv \end{bmatrix}$

b) nullity of $A = dim(A) - rank(A)$

$= n - 1$

c) $null(A) = \{x \in \mathbb{R}^n \mid A\vec{x} = 0\} = \{x \in \mathbb{R}^n \mid u_i \vec{v}\vec{x} = 0\}$ since $\vec{v}$ is nonezero

$\therefore \{x \in \mathbb{R}^n \mid \vec{v}\vec{x} = 0\}$

② prove $U = \{u_1, u_2, \ldots, u_n\}$ of non-zero pairwise-orthogonal vector in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.

Let $A = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$ and $A^T = \begin{bmatrix} - u_1 - \\ - u_2 - \\ \vdots \\ - u_n - \end{bmatrix}$

$\therefore A^TA = \begin{bmatrix} u_1u_1 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2u_2 & \cdots & u_2u_n \\ \vdots & & & \vdots \\ u_n u_1 & u_nu_2 & \cdots & u_nu_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix} = I$

since $u_i \cdot u_j = 0$ for $i \neq j$

$\therefore A$ has pivot in every colum.

$\therefore U = \{u_1, u_2, \ldots, u_n\}$ form a basis in $\mathbb{R}^n$.

4. linear Transformation

Lecture Notes

- if $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{v})=rT(\vec{v})$
  then a function $T: \mathbb{R}^n \longmapsto \mathbb{R}^m$ is a linear transformation $\forall \vec{v},\vec{u} \in \mathbb{R}^n$
  $\mathbb{R}^n$ is the domain of T $\mathbb{R}^m$ is the codomain of T
- if $W \subset \mathbb{R}^n$ then $T[W]=\{T(\vec{w}) \mid \vec{w} \in W\}$
- The range of T is $T[\mathbb{R}^n]=\{T(\vec{v}) \mid \vec{v} \in \mathbb{R}^n\}$
- $T^{-1}[W]=\{\vec{v} \in \mathbb{R}^n \mid T(\vec{v}) \in W\}$
- kernal of T: $T^{-1}[\{0\}]=\{v \in \mathbb{R}^n \mid T(v)=0\}$
- $T(r_1v_1+r_2v_2+\cdots+r_nv_n)=r_1T(v_1)+\cdots+r_nT(v_n)$
- $T(0)=0'$ $0 \in \mathbb{R}^n$ $0' \in \mathbb{R}^m$
- if $T: \mathbb{R}^n \to \mathbb{R}^m$ and $B=\{b_1,b_2,\dots,b_n\}$ is a basis for $\mathbb{R}^n$, and $v \in \mathbb{R}^n$
  then $T(\vec{v})$ is determined by $T(b_1), T(b_2), \dots, T(b_n)$
- if $T: \mathbb{R}^n \to \mathbb{R}^m$, then $T(\vec{x})=A\vec{x}$ $\forall x \in \mathbb{R}^n$

$$A=\begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$

- $\ker(T)=\text{null}(A)$ $\text{range}(T)=\text{col}(A)$
  A is the standard matrix representation of T
- $\text{rank}(T)=\dim(\text{col}(A))=\text{rank}(A)$
- $\text{nullity}(T)=\dim(\ker(T))=\text{nullity}(A)$
- T is invertible $\Leftrightarrow$ $T: \mathbb{R}^n \to \mathbb{R}^n$ or A is invertible
- $\text{rank}(T)+\text{nullity}(T)=\dim(\text{domain})$ of $T=\dim(\text{domain of } T)$
- one to one: $\text{kernel}(T)=\text{null}(A)$
  H has pivot in every colum
- onto: $\text{rank}(A)=m$
  H has a pivot in every row
- isomophism: both $\Leftrightarrow A^{-1}$ exist
  $H \sim I$

Assigments

① Suppose $T:\mathbb{R}^2\to\mathbb{R}^2$ and $T([1,1])=[3,2]$, $T([2,3])=[7,7]$. Find $T[x,y])$

since $[1,1]$ and $[2,3]$ are basis for $\mathbb{R}^2$, then $\left[\begin{array}{cc|c}1&2&x\\1&3&y\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&3x-2y\\0&1&y-x\end{array}\right]$

$\therefore\ 3x-2y[1,1]+y-x[2,3]=[x,y]$

$\therefore\ T([x,y])=3x-2y\,T([1,1])+y-x\,T([2,3])$

$=3x-2y[3,2)+y-x[7,7]$

$=[2x+y,\ -x+3y]$

② $T:\mathbb{R}^4\to\mathbb{R}^3$ $T([x_1,x_2,x_3,x_4])=[x_2-3x_2-x_4,\ 6x_1+5x_2,\ x_3+2x_1]$ find A

$T([1,0,0,0])=[0,6,2]$

$T([0,1,0,0])=[1,5,0]$ $\quad A\begin{bmatrix}0&1&\\6&5&\cdots\\2&0&\end{bmatrix}$

⋮

③ $T:\mathbb{R}^n\to\mathbb{R}^m$ and $W\subset\mathbb{R}^n$, prove $T[W]$ is a subspace for $\mathbb{R}^m$

i) since $W\subset\mathbb{R}^n$ then W is none empty, let $\vec{v}\in W$ then $T(\vec{v})\in T[W]$ $\therefore$ $T[W]$ is not empty

ii) if $\vec{x},\vec{y}\in W$ then $\exists\ u,v\in T[W]$ s.t $T[x]=u$ and $T[y]=v$

since $\vec{x}+\vec{y}\in W$ then $T(x+y)\in T[W]$ $\therefore$ $u+v=T(x)+T(y)=T(x+y)\in T[W]$

⋮

④ $T:\mathbb{R}^3\to\mathbb{R}^3$ and $B=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is a basis for $\mathbb{R}^3$. $T(\vec{v}_1)=\begin{bmatrix}-2\\1\\1\end{bmatrix}$, $T(\vec{v}_2)=\begin{bmatrix}0\\1\\-1\end{bmatrix}$, $T(\vec{v}_3)=\begin{bmatrix}-2\\2\\0\end{bmatrix}$

a) Determine if $w=\begin{bmatrix}-6\\5\\0\end{bmatrix}$ is in range(T)

$\text{range}(T)=\text{span}\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix},\begin{bmatrix}-2\\2\\0\end{bmatrix}\right\}$

$\left[\begin{array}{ccc|c}-2&0&-2&-6\\1&1&2&5\\1&-1&0&0\end{array}\right]\sim\left[\begin{array}{ccc|c}1&1&2&5\\0&2&2&4\\0&0&0&-1\end{array}\right]$ inconsistent $\Rightarrow$ not in range of T

b) basis for range(T)?

$\begin{bmatrix}-2&0&-2\\1&1&2\\1&-1&0\end{bmatrix}\sim\begin{bmatrix}1&1&2\\0&2&2\\0&0&0\end{bmatrix}$

c) nullity(T)?

nullity(T) = nullity(A)

4. $a=[a_1,a_2]\in\mathbb{R}^2$ and $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(x)=\left(\frac{x\cdot a}{a\cdot a}\right)a$

a) Find $T([x,y])$

$$T([x,y])=\left(\frac{[x,y]\cdot[a_1,a_2]}{[a_1,a_2]\cdot[a_1,a_2]}\right)[a_1,a_2]=\frac{a_1x+a_2y}{a_1^2+2a_1a_2+a_2^2}[a_1,a_2]=\left[\frac{a_1(a_1x+a_2y)}{a_1^2+2a_1a_2+a_2^2},\ \frac{a_2(a_1x+a_2y)}{a_1^2+2a_1a_2+a_2^2}\right]$$

b) Prove T is a linear transformation

(i) let $\vec{u}=[u_1,u_2]$ and $\vec{v}=[v_1,v_2]$ WTS $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$

$$T(\vec{u}+\vec{v})=T([u_1+v_1,u_2+v_2])=\left(\frac{[u_1+v_1,u_2+v_2]\cdot[a_1,a_2]}{a_1^2+2a_1a_2+a_2^2}\right)[a_1,a_2]$$

$$=\left(\frac{(u_1+v_1)a_1+(u_2+v_2)\cdot a_2}{a_1^2+2a_1a_2+a_2^2}\right)[a_1,a_2]$$

where $T(\vec{u})+T(\vec{v})=\left(\frac{[u_1,u_2]\cdot[a_1,a_2]}{a_1^2+2a_1a_2+a_2^2}\right)[a_1,a_2]+\left(\frac{[v_1,v_2]\cdot[a_1,a_2]}{a_1^2+2a_1a_2+a_2^2}\right)[a_1,a_2]$

$$=\left(\frac{u_1a_1+u_2a_2}{a_1^2+2a_1a_2+a_2^2}\right)[a_1,a_2]+\left(\frac{v_1a_1+v_2a_2}{a_1^2+2a_1a_2+a_2^2}\right)[a_1,a_2]$$

$=$ ✓

c) find A

$T(e_1)=T[1,0]=\left(\frac{1}{a_1^2+a_2^2}\right)[a_1^2,a_1a_2]$

$T(e_2)=T[0,1]=\left(\frac{1}{a_1^2+a_2^2}\right)[a_1a_2\ \ a_1^2]$

$\therefore A=\left(\frac{1}{a_1^2a_2^2}\right)\begin{bmatrix}a_1^2 & a_1a_2\\ a_1a_2 & a_2^2\end{bmatrix}$

5. Determinants

Lecture Notes

- the determinant of $[3]$ is 3
- the determinant of $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is $a_1 b_2 - a_2 b_1$
- the determinat of $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is $a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$
  $= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$
- $a = [a_1, a_2, a_3]$ $b = [b_1, b_2, b_3]$
  then $a \times b = \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right]$
  $\begin{matrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$
- $a \times b$ is perpendicular to both a and b.
- parallelogram $\vec{a} = [a_1, a_2]$ $\vec{b} = [b_1, b_2]$ in $\mathbb{R}^2$ Area $= \left| \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|$
- parallelogram $\vec{a} = [a_1, a_2, a_3]$ $\vec{b} = [b_1, b_2, b_3]$ in $\mathbb{R}^3$ Area $= \|a \times b\|$
- parallelepipe $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$, $\vec{c} = [c_1, c_2, c_3]$
  Volume $= \left\| \begin{bmatrix} - \vec{a} - \\ - \vec{b} - \\ - \vec{c} - \end{bmatrix} \right\|$
- $\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & {}_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \underbrace{a_{11}}_{\text{entry}} \overbrace{a'_{11}}^{\text{cofactor}} + a_{12}a'_{12} + \cdots + a_{1n}a'_{1n}$
- $\det(A) = \det(A^T)$ $\quad \det(\ A) = r^n \det(A)$
- Elem. Row. Operation in terms of determinant
  1) $A \xrightarrow{R_i \leftrightarrow R_j} B$ $\det(B) = -\det(A)$
  2) A has two equal rows $\Rightarrow \det(A) = 0$
  3) $A \xrightarrow{R_i \leftrightarrow rR_i} B$ $\det(B) = r\det(A)$
  4) $A \xrightarrow{R_i \to R_i + rR_j} B$ $\det(B) = \det(A)$
- $\det(EA) = \det(E)\det(A)$
- if A contains proportional rows or columns, then $\det(A) = 0$
- A is invertible $\Leftrightarrow \det(A) \neq 0$
- $\det(AB) = \det(A)\det(B)$
- if A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
- if A is $n \times n$ then $(\mathrm{adj}(A))A = A(\mathrm{adj}(A)) = \det(A)I$

where $\mathrm{adj}(A) = (A')^T$ $A' = [a_{ij}']$

- if A is $n\times n$ and $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$$

- ☆ if $Ax = b$ is $n\times n$ and $\det(A)\neq 0$, then the unique solution $x = [x_1, x_2, \dots, x_n]$ is in the form: $\boxed{x_k = \frac{\det(B_k)}{\det(A)}}$ $k = 1,\dots,n$ where B_k is the matrix A with the kth column replaced by b

Assigments

① compute $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 3 & 3 & 3 & 9 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 2 & 0 \end{vmatrix}$

$$A = \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 3 & 3 & 9 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix}}_{A} \xrightarrow{\substack{R_2-3R_1 \\ R_3-R_1 \\ R_4-R_1}} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 3 & 6 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & 2 & -1 \end{bmatrix}}_{A_1} \xrightarrow{R_2 \leftrightarrow R_4} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & -3 & 3 & 6 \end{bmatrix}}_{A_2} \xrightarrow{\substack{R_3+2R_2 \\ R_4-3R_2}} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & -3 & 9 \end{bmatrix}}_{A_3} \xrightarrow{\substack{3R_3 \\ 5R_4}} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 15 & 3 \\ 0 & 0 & -15 & 45 \end{bmatrix}}_{A_4} \xrightarrow{R_4+R_3} \underbrace{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 15 & 3 \\ 0 & 0 & 0 & 48 \end{bmatrix}}_{A_5}$$

$|A| = |A_1|$ $|A_2| = -|A|$ $|A_3| = -|A|$ $|A_4| = -15|A|$ $|A_5| = -15|A|$

$|A_5| = (-15)(48)$

$\therefore |A| = 48$

② $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix}$ find $\mathrm{adj}(A)$ and A^{-1} $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$

$a_{11}' = 1\begin{vmatrix} 1 & 5 \\ -6 & 7 \end{vmatrix} = 37$ $a_{12}' = -\begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} = -10$ $a_{13}' = 2$

$a_{21}' = 9$ $a_{22}' = 3$ $a_{23}' = 0$

$a_{31} = 17$ $a_{32}' = -5$ $a_{33}' = 1$

$\therefore A' = \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}$ $\mathrm{adj}(A) = (A')^T = \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix}$

$\mathrm{adj}(A)\cdot A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \det(A)\cdot I$

$\therefore \det(A) = 3$ $\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix}$

- use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 - x_2 \quad &= 11 \\ x_2 + 4x_3 &= 3 \end{aligned}$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 11 \\ 3 \end{bmatrix}$

$|A| = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 2 - 12 = -10 \neq 0$

$B_1 = \begin{bmatrix} 0 & 1 & 1 \\ 11 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ $|B_1| = -30$ $\Rightarrow x_1 = \frac{-30}{-10} = 3$

$B_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 11 & 0 \\ 0 & 3 & 4 \end{bmatrix}$ $|B_2| = 50 \Rightarrow x_2 = \frac{50}{-10} = 5$ $\therefore x = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$

$B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 11 \\ 0 & 1 & 3 \end{bmatrix}$ $|B_3| = -20 \Rightarrow x_3 = \frac{-20}{-10} = 2$

③ if $\vec{u}, \vec{v} \in \mathbb{R}^3$

a) $u \cdot v = \|u\|\|v\| \Rightarrow u = kv,\ k>0$

b) $u \times v = 0 \Rightarrow u = kv$

c) $\|u \times v\| = \|u\|\|v\| \Rightarrow u \perp v$

d) $(u \times v) \cdot u = 0 \Rightarrow$ nothing

e) $(u \times v) \times u = 0 \Rightarrow u = kv$

④ show that $A_{n\times n} \Rightarrow \det(rA) = r^n \det(A)$

Let B be rA $\quad |B| = r|A_{n-1}| = r^2|A_{n-2}| = r^n|A|$

⑤ if $A_{4\times4}$ and $B_{4\times4}$ and $\det(A)=3$ $\det(B)=5$ what is the determinant

of $(3A)^{-1}$ $\quad 3(B^{-1})$ $\quad (2A)B^{-1}$ $\quad (\frac{1}{5}B)A$ $\quad (3A^{-1})$

$$\det((3A)^{-1}) = \frac{1}{\det(3A)} = \frac{1}{3^4\det(A)} = \frac{1}{3^5}$$

$$\det(3(B^{-1})) = 3^4\det(B^{-1}) = 3^4\frac{1}{\det(B)} = \frac{3^4}{5}$$

$$\det((2A)B^{-1}) = \det(2A)\det(B^{-1}) = 2^4\det(A)\frac{1}{\det(B)} = \frac{3\cdot 2^4}{5}$$

$$\det(\tfrac{1}{5}B)A = \det(\tfrac{1}{5}B)\det(A) = (\tfrac{1}{5})^4 5\times 3$$

$$\det(3A^{-1}) = 3^4\frac{1}{\det(A)} = 3^3$$

⑥ Let $U_{n\times n}$ s.t $U^TU = I$ show $|\det(U)|=1$

since $\det(U) = \det(U^T)$ then $1 = \det(I) = \det(U^TU) = \det(U^T)\det(U) = (\det(U))^2$

$\therefore \det(U) = \pm1 \quad \therefore |\det(U)| = 1$

⑦ Let $A\vec{x} = \vec{b}$ be a system where all entries in A and $\vec{b}$ are integers.

If $|\det(A)|=1$ prove x are all integers

since $|\det(A)|=1 \Rightarrow \det(A) = \pm1 \neq 0$ $\therefore$ solution is unique

$x_i = \frac{\det(B_i)}{\det(A)} = \pm\det(B_i) \Rightarrow$ integers only

⑧ matrix summing each row and column is 0. prove determinant of this is 0

if $x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, then that is a nonzero solution to $A\vec{x} = \vec{0}$ $\therefore$ matrix is not invertible. $\therefore \det(A) = 0$.

6. Eigenvalues and Eigenvectors

Lecture Notes

- Let $A_{n\times n}$, if $A\vec{v}=\lambda\vec{v}$ then λ is eigenvalue of A and v is the eigenvectors of A
- λ is a solution to $|A-\lambda I|=0$
- characteristic polynomial of A: $p(\lambda)=|A-\lambda I|$
- eigenspace of λ: $E_\lambda=\{\vec{v}\in\mathbb{R}^n \mid A\vec{v}=\lambda\vec{v}\}$
- $E_\lambda=\text{null}(A-\lambda I)$
- $T(\vec{v})=\lambda\vec{v}$
- if $A^k\vec{v}=\lambda^k\vec{v}$, then λ^k is eigenvalue and $\vec{v}$ is eigenvector of A.
- $A^{-1}\vec{v}=\frac{1}{\lambda}\vec{v}$
- if A is invertible $\Leftrightarrow$ $\lambda=0$ is not an eigenvalue of A
- $P^{-1}AP$ is a diagonal matrix $\Rightarrow$ A is diagonalizable
- if $\vec{v}_1,\dots,\vec{v}_k$ are eigenvalue corresp. to distinct eigenvalue of A $\Rightarrow$ $\vec{v}_1,\dots,\vec{v}_k$ are linearly independent
- $A_{n\times n}$ A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.
- if $A_{n\times n}$ has n distinct eigenvalues $\Rightarrow$ A is diagonalizable
- if $A_{n\times n}$, $B_{n\times n}$ s.t $B=P^{-1}AP$ $\Rightarrow$ A and B are similar
  - $\det(A)=\det(B)$
  - A is invertible $\Leftrightarrow$ B is invertible
  - $\text{rank}(A)=\text{rank}(B)$
  - $\text{nullity}(A)=\text{nullity}(B)$
  - $\det(A-\lambda I)=\det(B-\lambda I)$
- $(\lambda-\lambda_i)^{a_i}$ is a factor of $p(\lambda)$, but $(\lambda-\lambda_i)^{a_i+1}$ is not $\Rightarrow$ a_i is the algebraic multiplicity of λ_i
- $\dim(E_{\lambda_i})$ is the geometric multiplicity of λ_i
- $GM(\lambda_i)\leq AM(\lambda_i)$
- A is diagonalizable $\Leftrightarrow$ $p(\lambda)$ is a product of linear factors and $AM(\lambda_i)=GM(\lambda_i)$
- $A=P^{-1}DP \Rightarrow A^k=P^{-1}D^kP$

• Carley-Hamilton Theorem

if $P(\lambda)=(2-\lambda)(\lambda^2-1)=-\lambda^3+2\lambda^2+\lambda-2=0$ (O matrix)

then $P(A)=-A^3+2A^2+A-2I=O$

$$-A^3+2A^2+A=2I$$

$$A(-A^2+2A+I)=2I$$

$$\frac{1}{2}A(-A^2+2A+I)=I$$

$$A^{-1}=\frac{1}{2}(-A^2+2A+I)$$

• Every symmetric matrix is diagonalizable

Assigments

① Determine if $A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}$ is diagonalizable. If it is, find P and D s.t $D=P^{-1}AP$

$$P(\lambda)=|A-\lambda I|=\begin{vmatrix}-\lambda&0&-2\\1&2-\lambda&1\\1&0&3-\lambda\end{vmatrix}=-\begin{vmatrix}0&-2\\0&3-\lambda\end{vmatrix}+(2-\lambda)\begin{vmatrix}-\lambda&-2\\1&3-\lambda\end{vmatrix}-\begin{vmatrix}-\lambda&0\\1&0\end{vmatrix}$$

$$=(2-\lambda)(-\lambda(3-\lambda)+2)=(2-\lambda)(\lambda^2-3\lambda+2)=(2-\lambda)(\lambda-2)(\lambda-1)=-(2-\lambda)^2(\lambda-1)$$

$$\therefore \lambda_1=2 \quad \lambda_2=1$$

For $\lambda=2$

$A-2I=\sim\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix}$ $E\lambda_1=sp\left(\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

For $\lambda=1$

$A-I=\sim\begin{bmatrix}1&0&2\\0&1&-1\\0&0&0\end{bmatrix}$ $E\lambda_2=sp\left(\begin{bmatrix}-2\\1\\1\end{bmatrix}\right)$

$\therefore a(\lambda_1)=g(\lambda_1)$ and $a(\lambda_2)=g(\lambda_2)$

$\therefore$ A is diagonalizable

$P=\begin{bmatrix}1&0&-2\\0&1&1\\-1&0&1\end{bmatrix}$ $P^{-1}AP=D=\begin{bmatrix}2&0&0\\0&2&0\\0&0&1\end{bmatrix}$

② if $A=\begin{bmatrix}2&0&0\\0&0&1\\0&1&0\end{bmatrix}$, find A^{100}

$$A^{100}=P^{-1}D^{100}P$$

$|A-\lambda|=(2-\lambda)(\lambda-1)(\lambda+1)\Rightarrow\lambda_1=2 \quad \lambda_2=1 \quad \lambda_3=-1$

For each λ we have that

$E\lambda_1=sp\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ $E\lambda_2=sp\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)$ $E\lambda_3=sp\left(\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\Rightarrow P=\begin{bmatrix}1&0&0\\0&1&1\\0&0&-1\end{bmatrix}$ and $D=\begin{bmatrix}2&0&0\\0&1&0\\0&0&-1\end{bmatrix}$ use $\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&1&0&1&0\\0&0&-1&0&0&1\end{array}\right]$ get $P^{-1}=\begin{bmatrix}1&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&-\frac{1}{2}\end{bmatrix}$

$$\therefore A^{100}=\begin{bmatrix}1&0&0\\0&1&1\\0&1&-1\end{bmatrix}\begin{bmatrix}2^{100}&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac{1}{2}&-\frac{1}{2}\\0&\frac{1}{2}&-\frac{1}{2}\end{bmatrix}=\begin{bmatrix}2^{100}&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

③ consider $x_0=1$ $x_1=1$ $x_2=7$ $x_3=13$ $x_4=55$

$x_{k+2}=6x_k+x_{k+1}$

$$\begin{aligned} x_{k+1}&=x_{k+1} \\ 6x_k+x_{k+1}&=x_{k+2} \end{aligned} \qquad \overbrace{\begin{bmatrix}0&1\\6&1\end{bmatrix}}^{A}\overbrace{\begin{bmatrix}x_k\\x_{k+1}\end{bmatrix}}^{y_k}=\overbrace{\begin{bmatrix}x_{k+1}\\x_{k+2}\end{bmatrix}}^{y_{k+1}}$$

$\therefore$ $Ay_k=y_{k+1}$ where $y_0=\begin{bmatrix}x_0\\x_1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ and $A\vec{y_0}=\vec{y_1}=\begin{bmatrix}1\\7\end{bmatrix}$

$\therefore A^k\vec{y_0}=\vec{y_k}$ (since $A\cdot\underbrace{A\vec{y_0}}_{\vec{y_1}}=\vec{y_2}$)

find A^{999} and we get $\vec{y_k}$

④ $x_0=1$, $x_1=1$, $x_2=7$, $x_3=25$, $x_4=103$

$x_{k+2}=3x_{k+1}+4x_4$

$$\underbrace{\begin{bmatrix}0&1\\4&3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_k\\x_{k+1}\end{bmatrix}}_{y}=\begin{bmatrix}x_{k+1}\\x_{k+2}\end{bmatrix}$$

$\therefore A^{27}y_0=y^{27}$

$y_0=\begin{bmatrix}1\\1\end{bmatrix}$

$$|A-\lambda I|=\begin{vmatrix}-\lambda&1\\4&3-\lambda\end{vmatrix}=-\lambda(3-\lambda)-4=-3\lambda+\lambda^2-4=\lambda^2-3\lambda-4$$
$$=(\lambda+1)(\lambda-4)$$

$\lambda_1=-1$ $\lambda_2=4$

For $\lambda_1=-1$ $A+I=\begin{bmatrix}1&1\\4&4\end{bmatrix}\sim\begin{bmatrix}1&1\\0&0\end{bmatrix}\Rightarrow E_{\lambda_1}=sp(\begin{bmatrix}1\\-1\end{bmatrix})$

For $\lambda_2=4$ $A-4I=\begin{bmatrix}-4&1\\4&-1\end{bmatrix}\sim\begin{bmatrix}4&-1\\4&-1\end{bmatrix}\sim\begin{bmatrix}4&-1\\0&0\end{bmatrix}\Rightarrow E_{\lambda_2}=sp(\begin{bmatrix}4\\1\end{bmatrix})$

$\therefore P=\begin{bmatrix}1&4\\-1&1\end{bmatrix}$ $P^{-1}=\frac{1}{5}\begin{bmatrix}1&-4\\1&1\end{bmatrix}$ $D=\begin{bmatrix}-1&0\\0&4\end{bmatrix}$

$$A^{27}=\begin{bmatrix}1&4\\-1&1\end{bmatrix}\begin{bmatrix}-1&0\\0&4^{27}\end{bmatrix}\begin{bmatrix}1&-4\\1&1\end{bmatrix}\frac{1}{5}$$
$$=\frac{1}{5}\begin{bmatrix}1+4^{28}&-4+4^{28}\\-1+4^{27}&4+4^{27}\end{bmatrix}$$

7. Complex Numbers

Lecture Notes

- $|z| = r = \sqrt{a^2+b^2}$
- $Arg(z) = \theta$
- $z = a+ib = r\cos\theta + ir\sin\theta$ — polar form of z
- $z_1 z_2 = |z_1||z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$
- $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$
- $\arg(z^n) = n\theta$
- $\overline{a+ib} = a-ib$
- $|z| = \sqrt{z\overline{z}}$

1. $\overline{\overline{z}} = z$ and $|z|^2 = z\overline{z}$.
2. $\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$
3. $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
4. $z^{-1} = \dfrac{\overline{z}}{|z|^2}$
5. $\overline{z_1/z_2} = \overline{z}_1/\overline{z}_2$
6. $Re(z) = \dfrac{z+\overline{z}}{2}$ and $Re(z) \leq |z|$
7. $|z_1 + z_2| \leq |z_1| + |z_2|$.

- $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$
- $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$

Assigments

① Find all solution to $z^8=1$

$$z^8=1=\cos(0)+i\sin(0)$$
$$=\cos(0+2k\pi)+i\sin(0+2k\pi)$$

$$z^{\frac{1}{8}}=1^{1/8}(\cos\left(\frac{2k\pi}{8}\right)+i\sin\left(\frac{2k\pi}{8}\right)$$

$k=0$ $\quad z_0=(\cos(0)+i\sin(0)=1$

$$z_1=\cos\left(\frac{2\pi}{8}\right)+i\sin\left(\frac{2\pi}{8}\right)=\cdots$$

$\cdots\cdots$

② find all complex number x such that x^4-1

$$x^4=1$$

$$z^4=1=\cos(0+2k\pi)+\sin(0+2k\pi)$$

$$z^{\frac{1}{4}}=\cos\left(\frac{2k\pi}{4}\right)+i\sin\left(\frac{2k\pi}{4}\right)$$

$$z_0=1$$

$$z_1=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)$$

$$z_2=\cos(\pi)+i\sin(\pi)$$

$$z_3=\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)$$

Review seminar

① Find all complex number s.t $x^4=-1$

$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$

and based on given $n=4, r=1, \theta=\pi$

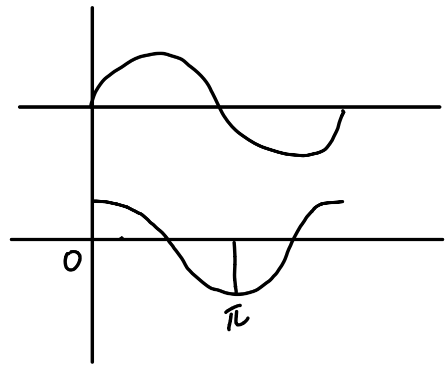
$\therefore z^{\frac{1}{4}} = 1\left(\cos\left(\frac{\pi+2k\pi}{4}\right) + i\sin\left(\frac{\pi+2k\pi}{4}\right)\right)$

$k=0 \quad z^{\frac{1}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$k=1 \quad z^{\frac{1}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$k=2 \quad z^{\frac{1}{4}} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$k=3 \quad z^{\frac{1}{4}} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$



②

$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ prove $a+d=\lambda_1+\lambda_2$

$p(\lambda)=|A-\lambda I| = \begin{vmatrix} a-\lambda & b\\ c & d-\lambda\end{vmatrix} = (a-\lambda)(d-\lambda)-bc$

$= ad - a\lambda - d\lambda + \lambda^2 - bc$

$= \lambda^2 - \lambda(a+d) - (ad+bc)$

since λ_1 and λ_2 are root of $p(\lambda)$

$\therefore p(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)$

$= \lambda^2 - \lambda(\lambda_1+\lambda_2) + \lambda_1\lambda_2$

$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ $\lambda_1\lambda_2$ prove $\det(A)=\lambda_1\lambda_2$

$P(\lambda) = \lambda^2 - \lambda(a+d) + (ad+bc)$

and $P(\lambda_1) = \lambda_1^2 - \lambda_1(a+d) + \underbrace{(ad+bc)}_{\det(A)}$

$= \lambda_1^2 - \lambda_1 - \lambda_1\lambda_2 - \det(A)$